Lessons 24-26: Optimization

How to Solve Optimization Problems

1. Sketch, if possible. Identify all given quantities and all quantities to be determined.
2. Write a primary equation that is to be maximized or minimized (the **objective function**).
3. Reduce the primary equation to ONE independent variable using the **constraint equations**.
4. Find the domain of the primary equation.
5. Use calculus to find the maximum or minimum.
6. Answer the question.

1. Find two positive numbers such that the first number plus twice the second number is a minimum and the product of the two numbers is 36.

1. $xy = 36$
2. Minimize $A = x + 2y$
3. $xy = 36 \Rightarrow x = \frac{36}{y}$ $(y \neq 0)$

   $\Rightarrow A = \frac{36}{y} + 2y$
4. $x, y > 0$
5. $A' = -\frac{36}{y^2} + 2 \overset{\text{set}}{=} 0$

   $\frac{-36 + 2y^2}{y^2} = 0$

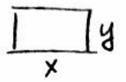
   $y^2 = 18$

   $y = \sqrt{18}$

   $x = \frac{36}{\sqrt{18}} = 2\sqrt{18}$

   A': $-$, $+$
   y: 0, $\sqrt{18}$
6. $y = \sqrt{18}$, $x = 2\sqrt{18}$

2. Find the dimensions of a rectangle with area 12cm^2 and a minimum perimeter.

1. 

$xy = 12$

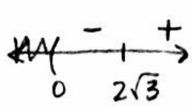
2. $P = 2x + 2y$

3. $xy = 12 \Rightarrow y = \frac{12}{x}$

$\Rightarrow P = 2x + \frac{24}{x}$

4. $x, y > 0$

5. $P' = 2 - \frac{24}{x^2} \overset{\text{set}}{=} 0$



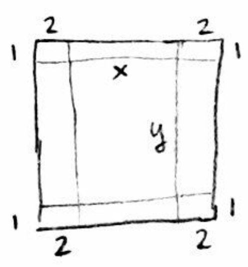
$\frac{2x^2 - 24}{x^2} = 0$

$x = \sqrt{12} = 2\sqrt{3}$

$y = \frac{12}{2\sqrt{3}} = 2\sqrt{3}$

6. $x = y = \boxed{2\sqrt{3}}$ (so, a square!)

3. Ada wants to create a piece of art using a canvas with area 18ft^2. She wants to leave a one foot margin between the edge the canvas and the artwork on top and bottom, and two feet on either side. What dimensions of the canvas maximize the painted area?

1. 

$(x+4)(y+2) = 18$

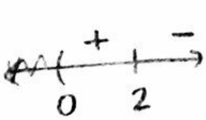
2. Maximize $A = xy$

3. $y = \frac{18}{x+4} - 2 \Rightarrow A = x\left(\frac{18}{x+4} - 2\right)$

$= \frac{18x}{x+4} - 2x$

4. $x, y > 0$

5. $A' = \frac{(x+4)(18) - 18x}{(x+4)^2} - 2$



$= \frac{72 - 2(x+4)^2}{(x+4)^2} \overset{\text{set}}{=} 0$

$(x+4)^2 = 36$

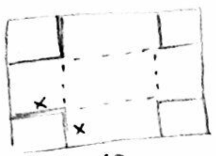
$x + 4 = +6$

$x = 2, \; y = 1$

6. Dimensions of canvas $= 6$ ft $\times$ 3 ft

4. A piece of cardboard is 24 inches by 48 inches. A square is to be cut from each corner and the sides folded up to make an open-top box. What is the maximum possible volume of the box?

1.



2. Maximize $V = (48-2x)(24-2x)x$

$= (48-2x)(24x-2x^2)$

4. $0 < x < 12$

5. $V' = -2(24x-2x^2)+(48-2x)(24-4x)$

$= -48x + 4x^2 + 1152 - 192x - 48x + 8x^2$

$= 12x^2 - 288x + 1152$

$= 12(x^2 - 24x + 96)$

$x = \frac{24 \pm \sqrt{24^2 - 4(96)}}{2} = \frac{24 \pm \sqrt{192}}{2} = \frac{24 \pm 8\sqrt{3}}{2} = 12 \pm 4\sqrt{3}$ ($12 + 4\sqrt{3}$ not in domain)

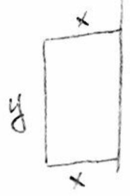
$+ \quad -$

$0 \quad 12 - 4\sqrt{3} \quad 12$

6. $V = (48 - 2(12-4\sqrt{3}))$
$\cdot (24 - 2(12 - 4\sqrt{3}))$
$\cdot (12 - 4\sqrt{3})$
$= (24 + 8\sqrt{3})(8\sqrt{3})(12 - 4\sqrt{3})$
$\approx \boxed{2660.4 \text{ in}^3}$

5. You have 8L feet of fence to make a rectangular exercise pen for your dog alongside the wall of your house. (L is a positive constant.) The wall of the house will bound one side of the pen. What is the largest possible area of the pen?

1.



$2x + y = 8L$

2. Maximize $A = xy$

3. $A = x(8L - 2x)$
$= 8Lx - 2x^2$

4. $0 < x < 4L$, $0 < y < 8L$

5. $A' = 8L - 4x \overset{set}{=} 0$ (L is a constant!)

$x = 2L$

$y = 4L$

6. $A = 2L(4L) = \boxed{8L^2 \text{ ft}^2}$

Check:

A' $+ \quad -$

$x \quad 0 \quad 2L \quad 8L$

$x = L$: $8L - 4L = 4L > 0$

$x = 3L$: $8L - 12L = -4L < 0$

OR

$A'' = -4 < 0$

so we have a max at $x = 2L$

6. Ted wants to construct a cylindrical flower pot ~~with height three times the radius~~. He has 60ft^2 of metal. What is the volume of the largest flower pot he can build?

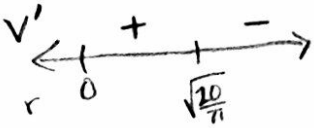
1.  $2\pi r h + \pi r^2 = 60$

2. Maximize $V = \pi r^2 h$

3. $h = \dfrac{60 - \pi r^2}{2\pi r} \Rightarrow V = \cancel{\pi} r^{\cancel{2}} \left(\dfrac{60 - \pi r^2}{2\cancel{\pi}\cancel{r}}\right)$
$= \frac{1}{2}(60r - \pi r^3)$

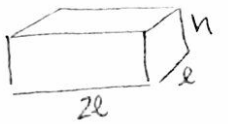
4. $r, h > 0$

5. $V' = \frac{1}{2}(60 - 3\pi r^2) \overset{\text{set}}{=} 0$
$r^2 = \frac{20}{\pi}$
$r = \sqrt{\frac{20}{\pi}}$



6. $V = \frac{1}{2}\left(60\sqrt{\frac{20}{\pi}} - \pi\left(\frac{20}{\pi}\right)^{3/2}\right) \approx \boxed{50.46 \text{ ft}^3}$

7. A moving box with width twice the length needs to have a volume of 32ft^3. Find the smallest possible amount of material needed.

1. 

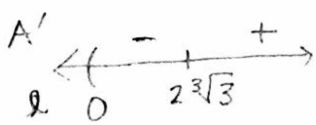
$2l^2 h = 32$

2. Minimize $A = 2(2l^2) + 2(lh) + 2(2lh)$
$= 4l^2 + 6lh$

3. $h = \dfrac{32}{2l^2} = \dfrac{16}{l^2} \Rightarrow A = 2l^2 + 6l\left(\frac{16}{l^2}\right)$
$= 2l^2 + \frac{96}{l}$

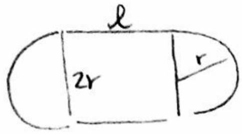
4. $l, h > 0$

5. $A' = 4l - \frac{96}{l^2} \overset{\text{set}}{=} 0$
$\dfrac{4l^3 - 96}{l^2} = 0$
$l = \sqrt[3]{\frac{96}{4}} = 2\sqrt[3]{3}$



6. $A = 2(2\sqrt[3]{3})^2 + \dfrac{96}{2\sqrt[3]{3}} \approx \boxed{49.92 \text{ ft}^3}$

8. Meriel wants to construct a circular table with a rectangular leaf that can be added in the middle. She wants the perimeter of the table (with the leaf in) to be 12 feet. What dimensions maximize the size of the leaf?

1. 

$P = 2\pi r + 2\ell = 12$

2. Maximize $A = 2r\ell$

3. $\ell = 6 - \pi r \Rightarrow A = 2r(6 - \pi r)$

$= 12r - 2\pi r^2$

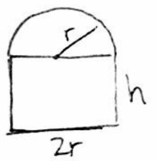
4. $r, \ell > 0$

5. $A' = 12 - 4\pi r \overset{\text{set}}{=} 0$ $\qquad A'' = -4\pi < 0$

$r = \frac{3}{\pi}$ $\qquad$ ∩ so max

6. $\boxed{r = \frac{3}{\pi} \text{ ft}, \ \ell = 3 \text{ ft}}$

9. Vincent plans to build a dog house whose cross section is a rectangle surmounted by a semicircle. If he wants the girth (perimeter of the cross section) to be 16 feet, what is the largest cross sectional area possible?

1. 

$P = 2r + 2h + \pi r = 16$

2. Maximize $A = 2rh + \frac{1}{2}\pi r^2$

3. $h = 8 - (1 + \frac{\pi}{2})r \Rightarrow A = 2r(8 - (1 + \frac{\pi}{2})r) + \frac{1}{2}\pi r^2$

$= 16r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$

$= 16r - 2r^2 - \frac{1}{2}\pi r^2$

4. $r, h > 0$

5. $A' = 16 - 4r - \pi r \overset{\text{set}}{=} 0$ $\qquad A'' = -4 - \pi < 0$

$r = \frac{16}{4+\pi}$ $\qquad$ ∩ so max

6. $A = 16\left(\frac{16}{4+\pi}\right) - 2\left(\frac{16}{4+\pi}\right) - \frac{1}{2}\pi\left(\frac{16}{4+\pi}\right)^2$

$\approx \boxed{17.92 \text{ ft}^2}$

10. Coke needs to make a cylindrical can that can hold precisely 500 cm^3 of liquid. Find the dimensions of the can that will minimize the cost.
The volume of a cylinder is $\pi r^2 h$ and the surface area is $2\pi rh + 2\pi r^2$ where r is the radius and h is the height.

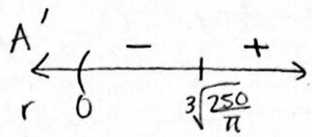
1.  $500 = \pi r^2 h$

2. Minimize $A = 2\pi rh + 2\pi r^2$

3. $h = \frac{500}{\pi r^2} \Rightarrow A = 2\cancel{\pi}\cancel{r}\left(\frac{500}{\cancel{\pi} r^{\cancel{2}}}\right) + 2\pi r^2$
$= \frac{1000}{r} + 2\pi r^2$

4. $r, h > 0$

5. $A' = \frac{-1000}{r^2} + 4\pi r \overset{\text{set}}{=} 0$

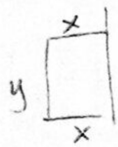


$\frac{-1000 + 4\pi r^3}{r^2} = 0$

$r = \sqrt[3]{\frac{250}{\pi}} = 5\sqrt[3]{\frac{2}{\pi}}$

6. $\boxed{r = 5\sqrt[3]{\frac{2}{\pi}},\ h = \frac{500}{\pi\left(5\sqrt[3]{\frac{2}{\pi}}\right)^2} = \frac{20}{\sqrt[3]{2\pi}}}$ cm

11. You do more reading and discover that the optimal size of an exercise pen is 40 ft^2. What is the minimum amount of fence you can use to create the pen, if you're still bounding one side of the pen with the wall of your house?

1. 

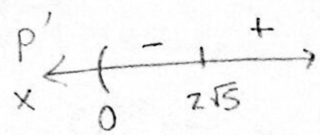
$xy = 40$

2. Minimize $P = 2x + y$

3. $y = \frac{40}{x} \Rightarrow P = 2x + \frac{40}{x}$

4. $x, y > 0$

5. $P' = 2 - \frac{40}{x^2} \overset{\text{set}}{=} 0$



$\frac{2x^2 - 40}{x^2} = 0$

$x = \sqrt{20} = 2\sqrt{5}$

6. $P = 2(2\sqrt{5}) + \frac{40}{2\sqrt{5}}$
$= 4\sqrt{5} + 4\sqrt{5} = \boxed{8\sqrt{5}\text{ ft}}$

12. Find the point on $f(x) = x^2$ closest to the point $(1, 5)$.

1. 

2. Minimize $d = \sqrt{(x-1)^2 + (x^2-5)^2}$ (or $D = (x-1)^2 + (x^2-5)^2$)
3. done!
4. $(-\infty, \infty)$
5. $D' = 2(x-1) + 2(x^2-5)(2x)$

$= 2x - 2 + 4x^3 - 20x$

$= 4x^3 - 18x - 2 \overset{set}{=} 0$

D': $-$ $+$ $-$ $+$

x: -2.06 $-.1142$ 2.1748

Calc $\Rightarrow x \approx -2.06, -.1142, 2.1748$

$d(-2.06) \approx 3.152$

$d(2.1748) \approx 1.20 \leftarrow$ smallest!

6. $(2.1748, 2.1748^2) = \boxed{(2.1748, 4.7298)}$

13. Ted has decided to make a cylindrical flower pot out of two types of metal. The metal for the sides costs \$15 per square foot and the metal for the base costs \$10 per square foot. Ted has \$30. What are the dimensions of the largest flower pot he can build?

1. 

$A_{base} = \pi r^2 \Rightarrow Cost_{base} = 10\pi r^2$

$A_{sides} = 2\pi r h \Rightarrow Cost_{sides} = 30\pi r h$

$10\pi r^2 + 30\pi r h = 30$

2. Maximize $V = \pi r^2 h$
3. $h = \frac{30 - 10\pi r^2}{30\pi r} \Rightarrow V = \cancel{\pi} r^{\cancel{2}} \left(\frac{30 - 10\pi r^2}{30 \cancel{\pi} \cancel{r}} \right)$

$= r - \frac{1}{3}\pi r^3$

4. $r, h > 0$
5. $V' = 1 - \pi r^2 \overset{set}{=} 0$

$r = \sqrt{1/\pi}$

V': $+$ $-$

r: 0 $\sqrt{1/\pi}$

or $V'' = -2\pi r$

$V''(\sqrt{1/\pi}) < 0$

max

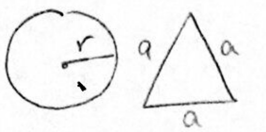
6. $\boxed{r = 1/\sqrt{\pi}, \; h = \frac{30 - 10}{30\sqrt{\pi}} = \frac{2}{3\sqrt{\pi}} \text{ ft}}$

14. A company estimates it will sell $q = 1500 - 300p$ units if each unit costs p dollars. If it costs them 1 dollar to make each unit, how many units should they make to maximize profit?
Revenue = Selling price * Units sold;
Profit = Revenue - Cost

1. Revenue $= (1500-300p)p$
   Profit $= (1500-300p)p - 1(1500-300p)$
2. Maximize $Pr = (1500-300p)(p-1)$
3. Done!
4. $p > 0$
5. $Pr' = -300(p-1) + (1500-300p)(1)$
   $= -300p + 300 + 1500 - 300p$
   $= -600p + 1800 \overset{set}{=} 0$
   $p = 3$

   $Pr'' = -600 < 0$
   $\frown$ max
6. units sold $= q = 1500 - 300(3) = \boxed{600 \text{ units}}$

15. The sum of the perimeters of a circle and equilateral triangle is 6 feet. What radius of the circle minimizes the total area? (Hint: The area of a equilateral triangle is $A = \frac{\sqrt{3}}{4}a^2$.)

1. 

   $2\pi r + 3a = 6$
2. Minimize $A = \pi r^2 + \frac{\sqrt{3}}{4}a^2$
3. $a = 2 - \frac{2}{3}\pi r \Rightarrow A = \pi r^2 + \frac{\sqrt{3}}{4}\left(2 - \frac{2}{3}\pi r\right)^2$
4. $a, r \geq 0$
5. $A' = 2\pi r + \frac{\sqrt{3}}{2}\left(2 - \frac{2}{3}\pi r\right)\left(-\frac{2}{3}\pi\right)$
   $= 2\pi r - \frac{2}{\sqrt{3}}\pi + \frac{2}{3\sqrt{3}}\pi^2 r \overset{set}{=} 0$
   $\left(2\pi + \frac{2}{3\sqrt{3}}\pi^2\right)r = \frac{2}{\sqrt{3}}\pi$
   $r = \frac{2\pi}{2\sqrt{3}\pi + \frac{2}{3}\pi^2} \cdot \frac{3}{3} = \frac{6\pi}{6\sqrt{3}\pi + 2\pi^2} = \boxed{\frac{3}{3\sqrt{3} + \pi}}$

   $A'' > 0$